LEMMA MATHEMATICS TASKS THAT PROMOTE HIGHER ORDER THINKING

3 Volume and Scale

TEACHER MANUAL

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The LEMMA series:
Learning Encounters with
Meta-Mathematical Activities

United Nations
Educational, Scientific and
Cultural Organization

New Zealand National
Commission for UNESCO
Te Kōmihana Matua o Aotearoa mō UNESCO

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LEMMA series overview

School mathematics curricula are often criticised as being “a mile wide and an inch deep”, meaning they cover a large range of topics, but only superficially. This can be frustrating for teachers who see mathematics as a joyful, captivating, and aesthetic experience, and wish to empower their students to use mathematical concepts flexibly and productively in contexts within and beyond the mathematics classroom.

The LEMMA series is a collection of booklets that contain sequences of *metamathematical* activities, which invite students to go beyond memorising and applying rules and skills, to thinking, working, and communicating with intriguing mathematical structures on a higher level. Topics in the series so far are shown below.

Topics and year levels	Link to New Zealand Curriculum
1 Equivalent Proportions (Year 9)	• Reason with linear proportions (*Level 5*)
2 Mixing Ratios (Year 10)	• Use rates and ratios (*Level 5*)
3 Volume and Scale (Year 11)	• Calculate volumes, including prisms, pyramids, cones, and spheres, using formulae (*Level 6: GM6-3*) • Recognise when shapes are similar and use proportional reasoning to find an unknown length (*Level 6: GM6-5*)
4 Circle Geometry (Year 11)	• Deduce and apply the angle properties related to circles (*Level 6: GM6-4*)
5 Graphical Antiderivatives (Years 12 & 13)	• Sketch the graphs of functions and their gradient functions and describe the relationship between these graphs (*Level 7: M7-9*) • Choose and apply a variety of differentiation, integration, and anti-differentiation techniques to functions and relations, using both analytical and numerical methods (*Level 8: M8-11*)

Each sequence of *metamathematical* activities follows the same structure:

- The sequence begins with a challenging open-ended activity, in which teams of students design a complex mathematical product
- Follow up activities invite students to critique and manipulate the mathematical structures underlying the initial activity from multiple perspectives
- Formative assessments tasks encourage students to extend their knowledge to new contexts.

The activities were developed and tested by a team of teachers, researchers, teacher educators, and mathematicians working in industry, with support from the Teaching and Learning Research Initiative and the University of Auckland. The writing of the LEMMA series was made possible through a Beeby Fellowship, jointly administered by UNESCO and New Zealand Council of Educational Research. We wish to express our heartfelt thanks to these organisations for their support.

Each set of tasks consists of:

- A **Teacher Manual** with questions, answers and notes on each task.
- A **Student Booklet** containing information students require for each task. This is a reusable booklet and should not be written in. There should be one booklet for each student in the class.
- A booklet of **Answer Sheets** for students to write the answers in. The sheets for each task can be photocopied from the booklet or downloaded for free from www.nzcer.org.nz/LEMMA.

Quick start guide

Start here

Warm-up task

Task 1: The snapper problem

Students create a team solution that expresses their understanding of how changes in length correspond to changes in volume. All other tasks build on this one, so it is important for students to work on this task before attempting any of the others.

Choose a sequence of follow ups*

Conceptual understanding activities	Conceptual understanding & Mathematical communication activities	Mathematical communication activities
Task 2	Task 2	Task 6
Task 3	Task 3	Task 7
Task 4	Task 4	
Task 5	Task 5	
	Task 6	
	Task 7	

End with formative assessment*

Task 8	Task 8	Task 8

* Or create your own sequence of follow ups and formative assessments, according to your students' learning needs

Warm-up task

Overview: The goal of this warm-up is to prepare students for Task 1 (the snapper problem), by engaging them in the marine animals context and mathematical concepts. It takes 10 minutes, and can be done with the whole class either the day before the snapper problem, or at the beginning of the class session when the snapper problem is implemented. The benefit of doing the warm-up the day before is that it will give students more time to work on their written letters for the snapper problem, and may leave some room for student presentations at the end. However, it also works well if the warm-up is presented on the same day as the snapper problem, as long as the warm-up is completed within 10 minutes to leave enough time for the task.

Many marine organisms grow proportionally

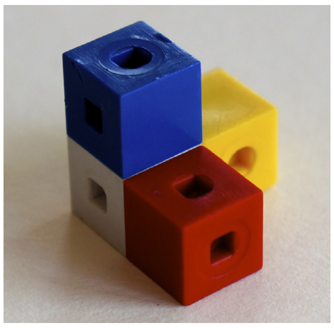
The baby seahorse is a miniature version of his father.

Humans do not grow proportionally. A baby human has shorter legs and arms compared to their head, whereas an adult human has longer legs and arms compared to their head.

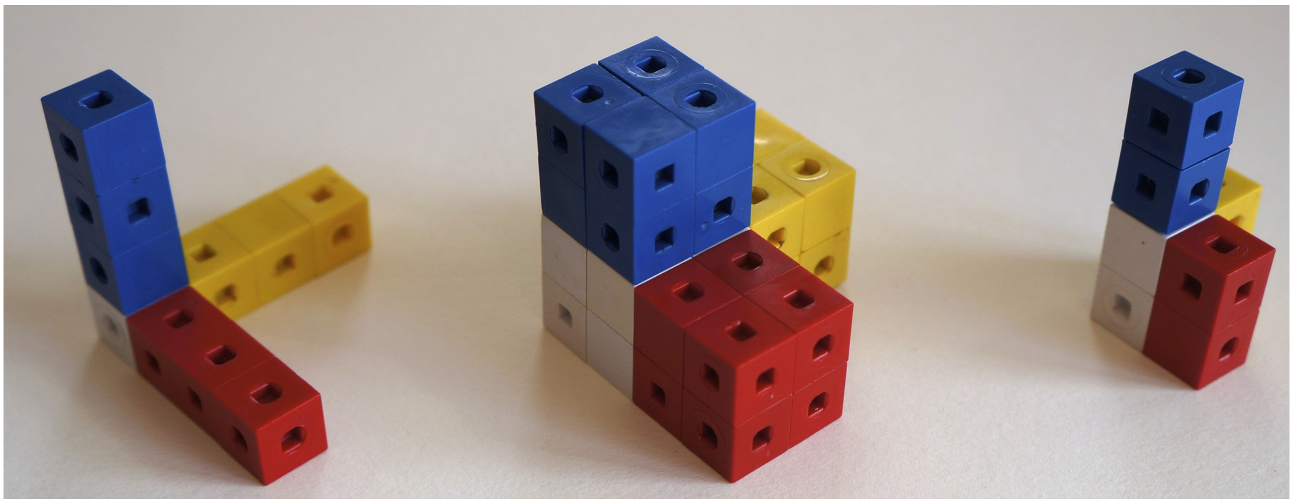
This is Squidley

Squidley lives in the sea, and like many other marine animals, he grows proportionally.

When he grows up he will be twice as big as he is now.



1. What will Squidley look like? Choose one option from (A), (B) and (C).



(A) (B) (C)

Note: Introduce Squidley as another marine animal. If possible, hand out models of Squidley and the optional adult versions of Squidley that you have made with connecting cubes. It is very useful for students to have access to these physical models as they work on the Squidley warmup and the subsequent Snapper problem. It helps students visualise proportional growth occurring in three dimensions.

Ask students to raise their hands to see how many have chosen which models. Most students will choose between options (A) and (C). Use the difference in opinions to motivate a class discussion of what it means to be "twice as big" and "proportional". Encourage students to justify their choices and challenge other students' justifications:

- Those who argue in favour of (B) will say it is "proportional", or that it "looks right". They may refer to the four big cubes that are proportional to Squidley's four little cubes.
- Those who argue in favour of (C) will point out that it is "twice as big", since it has twice as many small cubes as Squidley.
- Students will point out that (B) is not "twice as big" in the sense of volume. Some may say it is "twice as big" in terms of dimensions – a 2x2x2 cube as opposed to a 1x1x1 cube.
- Students will point out that (C) is not "proportional", since it has only grown up in one direction – height, not in the other two directions.

The discussion is likely to lead to students revealing some of the following misconceptions/errors:

- Students may call the cubes “squares”, saying it’s a 2x2 square, when really, they mean a 2x2x2 cube.
- Students may talk mainly about height and width, without considering all three dimensions.
- Students will usually initially say there are 4 times as many cubes in the left model as there are in Squidley. Some might say “8”, some might say “6”, “7”, “10”, “12”. Don’t worry if they all agree that the answer is 8 – surprisingly, knowing this relationship for Squidley doesn’t translate into them immediately recognising the same relationship for the snapper problem!!!

2. Option (C) has twice as many cubes as young Squidley, but there’s something not quite right. What is it?

Answer: Option C is twice as tall as Squidley, but not twice as wide or twice as thick. Option C doesn’t look proportional to young Squidley

3. Option (B) looks proportional to young Squidley, but it isn’t twice as big. How many cubes are there in (B)? How many in the young Squidley?

Answer: Option B has 32 cubes, whereas young Squidley has only 4 cubes.

Task 1: The snapper problem

Overview: This open-ended task forms the basis for the rest of the booklet, and should not be skipped. Students should work in teams of three for 30 minutes to create a team solution. If there is time available, teams can present their methods to the rest of the class. Facilitate mathematical questioning from the audience, and enjoy the diversity of ideas students bring to the problem.

The Lovrich family and the Borich family went fishing together. They caught nine snapper. Zoe Borich caught the biggest one, which was 54 cm long.


Joe Borich took the job of dividing the fish up fairly between the two families so that they had the same amount of fish each. He gave himself the big snapper (as his daughter Zoe caught it), and said that it was worth two of the smaller fish (27 cm each). Peter Lovrich thought that the flesh from the big fish was probably worth more but decided not to say anything to avoid a scene.

Peter needs your help!

Your job is to work out a mathematical argument for deciding how many little fish the big fish is worth. **Write a letter** to Peter, describing your mathematical argument clearly. Using a diagram might help. Peter wants to be able to use your argument for future fishing trips, so explain in your letter how he can make your argument work for fish of any size.

Note: As the problem statement is quite long, it works well to have one or two students read it out loud, as students are more likely to pay attention when they know they might be called upon to read part of the problem statement.

Before students start working, ask the class three questions to check they know what they are being asked to do:

(1) Question: Who's your client?
Answer: Peter Lovrich

(2) Question: What is Peter's problem?
Answer: He thinks he's being ripped off.

(3) Question: What does she want you to give her?

Answer: A ***letter*** that describes an ***argument***, not just the answer, and the method ***needs to work for fish of any size.***

Students may suggest solutions that don't deal with volume, such as:

- "fillet the fish and share up the fillets"
- "weigh the fish and divide the weights".

You may point out to the class that there are practical obstacles to both of these solutions:

- it is illegal to fillet fish on board as the department of fisheries needs to be able to determine the fish's length, since it is illegal to catch fish that are too small.
- It is impractical weighing fish on board a rocking boat. It is certainly possible to weigh it later on, but Peter wants an argument that doesn't rely on having weighing equipment.

Students may point out that the legal limit for snapper is currently 30cm in some areas, not 27cm (it was raised in 2014). Acknowledge this, and explain that the problem was written before the limit was increased to 30cm.

Students who just draw two-dimensional diagrams of fish may be locked into thinking that one large fish is worth four small fish. Encourage students to use the cubes from the Squidley warmup to model the fish.

Here are some frequently asked questions about implementing these open-ended tasks

1. Should I provide resources (e.g. calculators, counters, etc.) for the students? How should I provide them?

- The physical models of Squidley and his possible adult instantiations are very useful resources for students. Try to ensure that each group has access to these models during the warmup and during the task.
- Try not to prioritise one piece of equipment at the beginning, as students might think they have to use it to find the "correct" answer.
- You can put resources on group tables at the beginning of the task, but emphasise that these are just a selection of tools they might find useful, they don't have to use them, and they can use other tools if they wish.
- Alternatively, you could have a collection of resources on a common table at the front of the class, with the same caveat.

2. What members of a group of students don't talk to each other?

- Use 1 minute of thinking time for the whole class before starting the task. After launching the problem statement, say to the class: "Take 1 minute to think about this problem in silence". This will give students an opportunity to come up with ideas before they are forced to share them.
- When interacting with small groups that aren't talking, ask them to share what they are thinking with the other students in the group, not just with you. E.g., say "Sam, please tell Julia and Tina what ideas you've had about this problem", not "Sam, tell me what you're thinking". Then encourage the other team mates to respond to each others' ideas: "Julia, what do you think of Sam's suggestion?"

- Observe nonverbal clues and make them explicit to start discussion. E.g., “Benny, you’re frowning. What does that mean? Tell the others in your team what you’re thinking”.
- Make it clear that it’s OK to disagree with each other. Tell them they should challenge each other, because that’s how their ideas will get better.

3. What if a group of students finish before the rest of the class?

- Ask if they have written their letter.
- If they have, ask if their letter is clear enough for the client to follow without needing to ask for help.
- If their letter is clear, and their solution is sound, invite them to present their solution to the rest of the class once everyone’s finished. Let them spend the remaining time preparing their presentation while the rest of the class finish writing their letters.

4. What if some students are stuck or on the wrong track, mathematically?

- Remember that it’s OK for students to experience being stuck. Sometimes it’s a necessary phase before a breakthrough.
- Try not to give them explicit solutions or hints. Instead, get them to test their method on some examples that will reveal the method’s flaws. For example, you could draw a square and ask, “Would your method work on that?”
- Instead of telling them the answer, you could suggest they use a resource that will help see the problem in a new way. For example, you could say, “Could you show your thinking using the Squidley blocks?”

5. What if one student wants to do it all?

- In some cases, one or two students may take charge to the extent that they take over the problem. As a consequence, other team members may lose interest as they feel their contributions are not important. When this happens, remind the team that they all need to agree on the final method.
- Ask other team members whether they agree with the student’s method, and whether they have other suggestions.
- Ask the team member(s) who have not contributed much to write up the final method.
- Tell the team they all need to be comfortable with explaining their method to the rest of the class.

6. What if some students are off task?

- Sometimes, taking a break can be a useful part of the modelling process. After a short break, students often return to the problem with fresh eyes, which can lead to a new way of interpreting the data.
- Are they off task because of communication issues, or because they have already finished, or because they are stuck?
- Remind the students they are responsible for producing a group letter.
- Remember that it’s OK if some groups don’t finish the task as well as you hoped. By struggling on the problem, they will be better prepared to appreciate good solutions that other groups present.

7. What if some students won't write a letter?

- Getting students to write can always be a challenge. Some ways to scaffold this process are:
  - Encourage them to write step-by-step instructions.
  - Encourage them to draw diagrams.
  - Provide a writing frame like the one shown below:

> Dear Peter,
>
> This is how you should explain to Joe how many small fish one large fish is worth.
>
> Step 1:
>
> Step 2:
>
> Draw a diagram to explain your method
>
> Demonstrate your method with different sized fish.
>
> Yours sincerely,

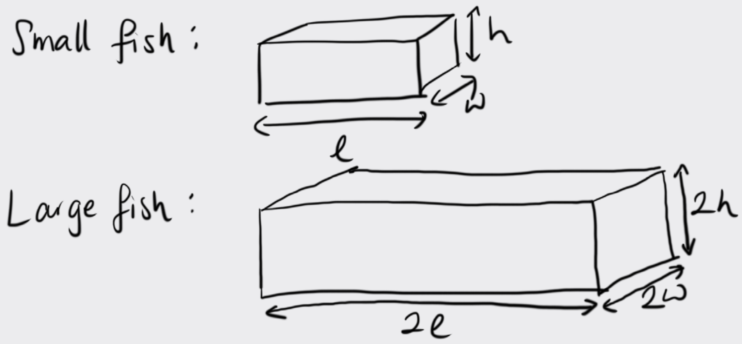
8. How do I mark the written communication of the letter?

Students may not have had much practice explaining mathematical methods in everyday English in the form of a letter. Some of the follow up tasks will be useful for enhancing students' written mathematical communication (e.g. Tasks 6 and 7), and the final assessment task (Task 8) is an opportunity for students to demonstrate how their written mathematical communication has developed. Guidelines on how to assess the written mathematical communication of Task 8 are given at the end of this booklet, and can certainly be applied to the snapper problem (Task 1) if you wish. However, it may be more productive to focus on strengthening students' written mathematical communication first before giving them an assessment, which is why we have not included the assessment guide in this task.

Sample answer:

Dear Peter,

One large fish is worth eight small fish if the large fish is twice the length, width and height of the small fish. Assume we can model the volume of a fish using a rectangular prism:



From these pictures, we can find general expressions for the volumes of the two fish.

Volume of the small fish: $V_S = l \times w \times h$

Volume of the large fish: $V_L = 2l \times 2w \times 2h$

$= 8l \times w \times h$

$= 8V_S$

Sincerely, ...